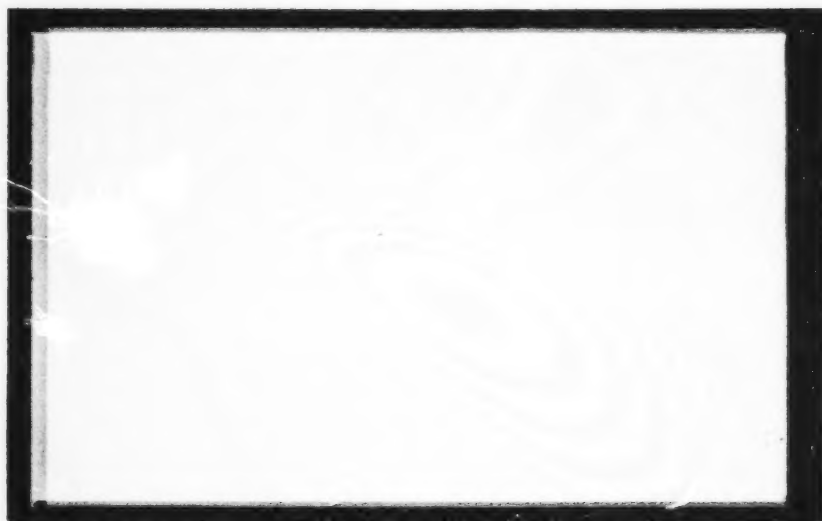


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**CENTRE DE RECHERCHE SUR LES TRANSPORTS**



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**OPTIMAL SOLUTIONS TO CAPACITATED  
MULTIDEPOT VEHICLE ROUTING PROBLEMS**

by

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### ABSTRACT

This paper provides an integer linear programming formulation and an exact algorithm for a class of multidepot vehicle routing problems. This formulation contains degree constraints, subtour elimination constraints, chain barring constraints and integrality constraints. Integrality is obtained by branch and bound; subtour elimination and chain barring constraints are first relaxed and only imposed when they are found to be violated. Computational results are reported.

### RESOLUTION OPTIMALE DE PROBLEMES DE TOURNEE AVEC PLUSIEURS DEPOTS ET CONTRAINTES DE CAPACITE

### RESUME

Cet article décrit une formulation par programmation linéaire en nombres entiers et un algorithme exact pour une famille de problèmes de tournée comportant plusieurs dépôts. Cette formulation contient des contraintes de degré, des contraintes d'élimination de sous-tours, des contraintes d'élimination de chaînes et des contraintes d'intégralité. On obtient l'intégralité par "branch and bound". Les contraintes d'élimination de sous-tours et de chaînes sont d'abord relaxées; on ne les introduit dans les problèmes que lorsqu'elles ne sont pas satisfaites. On présente des résultats numériques.

## 1. INTRODUCTION

The study of the travelling salesman problem (TSP) and of some of its extensions has led, in recent years, to major theoretical developments in the field of operational research and has provided new insights and solution techniques for a variety of location and vehicle routing problems encountered in real life situations. This paper examines an integer programming formulation for a class of TSP extensions called "multidepot vehicle routing problems".

We first introduce some notation and definitions. Let  $G = (N, E)$  be a graph where  $N = \{1, \dots, n\}$  is a set of nodes (cities) and where  $E$  is a set of undirected edges. With  $E$ , we first associate a symmetrical cost or distance matrix  $C^* = (c_{ij}^*)$  and then  $C = (c_{ij})$ , the matrix of shortest distances between nodes of  $N$ . If  $C^*$  satisfies the triangle inequality, i.e. if  $c_{ij}^* \leq c_{ik}^* + c_{kj}^*$  for all  $i, j, k \in N$ , then  $C = C^*$ . Note that  $C$  always satisfies the triangle inequality. Let  $R \subset N$  be a set of *depots*. There are  $m_r$  identical vehicles based at depot  $r$ , each with the same capacity  $D$  and a fixed cost  $f_r$ . To each node  $i$  of  $N - R$ , we associate a nonnegative requirement  $d_i$  ( $\leq D$ ).

The problem consists of determining how many vehicles are based at each selected depot and of establishing vehicle routes in such a way that

- (i) each route starts and ends at the same depot;
- (ii) all requirements are met exactly once by a vehicle (the same city  $i$  may be visited more than once if this saves distance but then, the requirement  $d_i$  of that city is satisfied only once by one vehicle);
- (iii) the sum of all requirements satisfied by any vehicle does not exceed  $D$ ;
- (iv) for each depot  $r$ ,  $m_r$  lies between a lower bound  $\underline{m}_r$  and an upper bound  $\bar{m}_r$ :  $1 \leq \underline{m}_r \leq m_r \leq \bar{m}_r$ ;

- (v) the total cost is minimized.

This problem is a generalization of many well known routing problems treated in the O.R. literature. In particular, it reduces to

- (i) P1: the TSP [1] if  $|R| = 1$ ,  $\sum_{r \in R} m_r = 1$  and  $\sum_{i \in N-R} d_i \leq D$ ;
- (ii) P2: the  $m$  salesmen problem [3] if  $|R| = 1$ ,  $\sum_{r \in R} m_r = m \geq 1$

$$\text{and } \sum_{i \in N-R} d_i \leq D;$$

- (iii) P3: the capacitated vehicle routing problem [4] if

$$|R| = 1, \sum_{r \in R} m_r = m \geq 1 \text{ and } \sum_{i \in N-R} d_i > D;$$

- (iv) P4: the multidepot multisalesmen problem [6] if

$$|R| > 1 \text{ and } \sum_{i \in N-R} d_i \leq D;$$

- (v) P5: the capacitated multidepot multisalesmen problem if

$$|R| > 1 \text{ and } \sum_{i \in N-R} d_i > D.$$

Our purpose is

- (i) to provide a unified integer linear programming formulation for P5 (which represents the largest class of multidepot vehicle routing problems treated in this paper) and to analyze some aspects of this formulation;
- (ii) to develop a suitable algorithm for this problem;
- (iii) to test the algorithm for P4 and P5.

## 2. FORMULATION

In addition to the notation already introduced, let us define

$$[t] = \begin{cases} \text{the smallest integer greater than or equal to } t \\ \text{if } t > 0 \\ 1 \text{ otherwise} \end{cases}$$

$x_{ij}$  : a variable indicating the number of times edge  $(i,j)$  is used in the optimal solution.  $x_{ij}$  is not defined if  $i \geq j$ , if  $i,j \in R$  or if  $d_i + d_j > D$ . In this paper,  $x_{ij}$  must be interpreted as  $x_{ji}$  whenever  $i > j$ .

P5 can be formulated as follows:

$$(P5) \text{ minimize } z = \sum_{i,j \in N} c_{ij} x_{ij} + \sum_{r \in R} f_r m_r$$

subject to

$$(1) \quad \sum_{i < k} x_{ik} + \sum_{k < j} x_{kj} = 2 \quad (k \in N-R)$$

$$(2) \quad \sum_{i < r} x_{ir} + \sum_{r < j} x_{rj} = 2m_r \quad (r \in R)$$

$$(3) \quad \sum_{i,j \in S} x_{ij} \leq |S| - \left\lceil \frac{\sum_{k \in S} d_k}{D} \right\rceil \quad (S \subseteq N-R, |S| \geq 3)$$

$$(4) \quad x_{i_1 i_2} + 3x_{i_2 i_3} + x_{i_3 i_4} \leq 4 \quad (i_1, i_4 \in R; i_2, i_3 \in N-R)$$

$$(5) \quad x_{i_1 i_2} + x_{i_{h-1} i_h} + 2 \sum_{i,j \in \{i_2, \dots, i_{h-1}\}} x_{ij} \leq 2h-5 \quad (h \geq 5; i_1, i_h \in R; \\ i_2, \dots, i_{h-1} \in N-R; \\ d_{i_2} + \dots + d_{i_{h-1}} \leq D)$$

$$(6) \quad 1 \leq m_r \leq m_r \leq m_r \quad (r \in R)$$

$$(7) \quad x_{ij} = 0, 1 \quad (i, j \in N-R)$$

$$x_{ij} = 0, 1, 2 \quad (i \text{ or } j \in R)$$

In this formulation, the  $m_r$ 's can be considered as constants fixed a priori, or as variables if their value is determined simultaneously with the optimal routes. Constraints (1) and (2) specify the degree of each node. Constraints (1) state that each city of  $N-R$  is visited exactly once. This will in fact be the case if  $C^*$  satisfies the triangle inequality; however, if it does not, the shortest path between  $i$  and  $j$  may well include other cities which will then be visited more than once, although this does not appear in the formulation. In this case, the requirement  $d_i$  of city  $i$  will be assigned only once to only one vehicle. Formulations for P2 and P3 [3,4] are obtained by removing constraints (4) and (5). P1 is a particular case of P2 where  $m_r = 1$ .

Constraints (3) ensure that no solution contains subtours disconnected from  $R$  and no route having a total weight exceeding  $D$ . Their derivation is provided in [4] and in [7]. They reduce to the standard TSP subtour elimination constraints [1]:

$$(8) \quad \sum_{i,j \in S} x_{ij} \leq |S| - 1 \quad (S \subseteq N-R, |S| \geq 3)$$

in all problems for which  $\sum_{i \in N-R} d_i \leq D$ .

Constraints (4) and (5) are *chain barring constraints*. They prohibit the formation of chains between two depots and hence ensure that each route starts and ends at the same depot. Their derivation is quite lengthy and is fully provided in [5].

### 3. ALGORITHM

The branch and bound algorithm developed to solve P5 can be summarized as follows.

Step 0: Obtain a first feasible solution by means of an appropriate heuristic. Let  $z^*$  be the cost of that solution.

Step 1: Select a subproblem from the list. (The first subproblem will include constraints (1) and (2) of P5 as well as the upper bounds on the variables.)

Step 2: Solve the subproblem using simplex (here we use the Land-Powell code [2]). Let  $\bar{z}$  be the cost of the optimal solution.

Step 3: If  $\bar{z} \geq z^*$ , proceed to step 9.

Step 4: The current solution contains

- (i) sets of nodes  $\{i_1, \dots, i_\ell\}$  ( $\ell > 1$ ) corresponding to chains  $(i_1, \dots, i_\ell)$  such that  $\{i_2, \dots, i_{\ell-1}\} \cap R = \emptyset$  if  $\ell > 2$  and for which all variables  $x_{i_1 i_2}, x_{i_2 i_3}, \dots, x_{i_{\ell-1} i_\ell}$  have been previously fixed at 1, and
- (ii) nodes not belonging to such chains (we define for each such node  $i$  a set  $\{i\}$ ).

For convenience, we refer to these sets of nodes  $S_k$  (corresponding to chains or single nodes) as *components*. Each  $S_k$  has an associated weight  $w(S_k)$  defined as

$$(9) \quad w(S_k) = \sum_{i \in S_k} d_i$$

Consider a component  $S_k$ . If it corresponds to a chain, let  $p_k$  and  $q_k$  be the end nodes of that chain; if it corresponds to a node  $i$ , let  $p_k = q_k = i$ . In the first case,  $x_{p_k q_k}$  can be forced to zero as long as  $p_k, q_k \in N-R$ . Now consider two components  $S_r$  and  $S_s$  and let  $i \in \{p_r, q_r\}$ ,  $j \in \{p_s, q_s\}$ . Then  $x_{ij}$  can be forced to zero if  $i, j \in N-R$  and  $w(S_r) + w(S_s) > D$ .

If step 4 has resulted in forcing any variable to zero, proceed to step 2.

Step 5: Check whether the current solution contains illegal subtours (i.e. subtours disconnected from  $R$  or having a total weight exceeding  $D$ ). If there are no illegal subtours, proceed to step 6. Otherwise, generate a type (3) constraint for each illegal subtour and proceed to step 2.

Step 6: Check whether the current solution contains illegal chains between depots. If there are no illegal chains, proceed to step 7. Otherwise, generate for each chain a chain barring constraint ((4) or (5)) and proceed to step 2.

Step 7: If the solution is integer, store it and set  $z^* = Z$ . Proceed to step 9. Otherwise, execute the next step.

Step 8: Select a fractional variable to branch upon and create new branches in the search tree according to the procedure described in [2]. Go to step 10.

Step 9: Back up in the search tree. This consists of modifying the level of the search tree (from  $\lambda'$  to  $\lambda$  where  $\lambda' \geq \lambda$ ) by freeing all variables either forced to zero or fixed at some integer value at levels  $\lambda, \lambda+1, \dots, \lambda'$ . Here we used the BACKUP subroutine of the Land-Powell code [2].

Step 10: Update the list of subproblems. If the list is empty, terminate and print the best solution. Otherwise, proceed to step 1.

#### 4. COMPUTATIONAL RESULTS

The algorithm was tested on a number of randomly generated problems for various values of the parameters. The  $c_{ij}$ 's were obtained as follows:  $n$  points were first generated in  $[0,100]^2$ . The distance between two points  $(x_i, y_i)$  and  $(x_j, y_j)$  was then defined as

$$(10) \quad c_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{\frac{1}{2}}$$

For P5, the  $d_i$ 's were generated according to a uniform distribution on  $[0,100]$ .  $D$  was then defined as

$$(11) \quad D = (1 - \alpha) \max_i \{d_i\} + \alpha \sum_{i=1}^n d_i$$

where  $\alpha$  is a parameter chosen in  $[0,1]$ . When  $\alpha$  is small, the vehicle capacity is small relative to the requirements and, with our approach, the occurrence of illegal subtours is more likely. Conversely, when  $\alpha$  is large, the probability that a single vehicle will suffice to visit many cities increases and less illegal subtours are thus likely to occur.

In all problems which were considered, the  $m_r$ 's were set equal to 1 and the  $\bar{m}_r$ 's, to infinity. Computational experience on P3 [4] has revealed that fixing the  $m_r$ 's a priori or imposing on them a tight upper bound reduces the growth of the search tree and thus yields easier problem on the whole. Similarly, using fixed costs  $f_r$  on the vehicles increases the number of subproblems which are dominated in the search tree and produces relatively easy problems. For these reasons, no upper bounds and no  $f_r$ 's were used.

The algorithm described in section 4 was coded in FORTRAN by adding subroutines to or by otherwise modifying the Land-Powell BB program [2]. All problems were solved on the University of Montreal CYBER 855 computer, using an FTN5 compiler. Problems "failed" when they could not be solved within 300 seconds or within the maximum allowed memory of 200 000<sub>8</sub> words.

Tables 1 and 2 report average results for the number of successful problems out of 3.

Our tests indicate that the problems can be solved optimally up to about 40 nodes in the case of P4 and up to about 25 nodes in the case of P5. (Note that P4 is a particular case of P5 with  $\alpha = 1.0$ .) The difficulty of the problems varies of course with the choice of the parameters. As a rule, computation times decrease as  $\alpha$  gets larger and increase with  $|R|$ . The column "maximum number of effective constraints" gives an idea of the difficulty of the problem: in problems such as the TSP and the m-TSP (P1 and P2), the values contained in this column hardly exceed  $n$ , i.e. the number of degree constraints (which tend to be effective throughout the algorithm - see for exam-

TABLE 1  
Computational results for P4

| $n$ | $ R $ | number of successful<br>problems out of 3 | time<br>(seconds) | number of simplex<br>iterations | maximum number of<br>effective constraints | total number of chain<br>barring constraints | total number of<br>subtour elimination<br>constraints |
|-----|-------|-------------------------------------------|-------------------|---------------------------------|--------------------------------------------|----------------------------------------------|-------------------------------------------------------|
| 20  | 2     | 3                                         | 2.88              | 100.7                           | 28.3                                       | 0.0                                          | 5.3                                                   |
|     | 5     | 3                                         | 2.44              | 80.3                            | 34.7                                       | 0.7                                          | 5.3                                                   |
|     | 8     | 3                                         | 4.24              | 99.3                            | 38.0                                       | 0.0                                          | 2.0                                                   |
| 30  | 2     | 3                                         | 10.37             | 172.0                           | 38.3                                       | 3.3                                          | 6.3                                                   |
|     | 5     | 2                                         | 50.78             | 601.5                           | 46.0                                       | 4.5                                          | 8.5                                                   |
|     | 8     | 1                                         | 6.07              | 100.0                           | 50.0                                       | 2.0                                          | 5.0                                                   |
| 40  | 2     | 3                                         | 122.20            | 766.0                           | 51.7                                       | 0.0                                          | 13.7                                                  |
|     | 5     | 0                                         | -                 | -                               | -                                          | -                                            | -                                                     |
|     | 8     | 1                                         | 20.82             | 147.0                           | 60.0                                       | 4.0                                          | 4.0                                                   |
| 50  | 2     | 2                                         | 190.00            | 880.5                           | 64.0                                       | 0.0                                          | 18.5                                                  |

TABLE 2

Computational results for P5

| $n$ | $ R $ | $\alpha$ | number of successful<br>problems out of 3 | time<br>(seconds) | number of simplex<br>iterations | maximum number of<br>effective constraints | total number of chain<br>barring constraints | total number of<br>subtour elimination<br>constraints |
|-----|-------|----------|-------------------------------------------|-------------------|---------------------------------|--------------------------------------------|----------------------------------------------|-------------------------------------------------------|
| 15  | 2     | 0.0      | 1                                         | 189.49            | 5577.0                          | 65.0                                       | 4.0                                          | 44.0                                                  |
|     |       | 0.25     | 0                                         | -                 | -                               | -                                          | -                                            | -                                                     |
|     |       | 0.50     | 3                                         | 14.64             | 389.0                           | 29.0                                       | 14.0                                         | 18.7                                                  |
|     |       | 0.75     | 3                                         | 7.21              | 256.3                           | 26.7                                       | 7.7                                          | 8.3                                                   |
|     |       | 1.00     | 3                                         | 0.96              | 60.3                            | 21.3                                       | 0.7                                          | 3.0                                                   |
|     | 5     | 0.0      | 2                                         | 43.14             | 1290.0                          | 32.5                                       | 15.0                                         | 17.0                                                  |
|     |       | 0.25     | 2                                         | 104.08            | 2314.0                          | 37.5                                       | 13.5                                         | 24.0                                                  |
|     |       | 0.50     | 3                                         | 118.51            | 2203.3                          | 33.0                                       | 19.7                                         | 11.7                                                  |
|     |       | 0.75     | 2                                         | 60.91             | 1490.0                          | 31.5                                       | 9.5                                          | 4.0                                                   |
|     |       | 1.00     | 3                                         | 41.31             | 1020.7                          | 30.3                                       | 7.3                                          | 3.3                                                   |
|     | 8     | 0.0      | 3                                         | 6.23              | 193.0                           | 32.7                                       | 3.7                                          | 8.0                                                   |
|     |       | 0.25     | 2                                         | 4.60              | 113.0                           | 35.7                                       | 2.0                                          | 1.0                                                   |
|     |       | 0.50     | 3                                         | 9.60              | 92.3                            | 33.3                                       | 1.3                                          | 1.0                                                   |
|     |       | 0.75     | 3                                         | 3.10              | 92.3                            | 33.3                                       | 1.3                                          | 1.0                                                   |
|     |       | 1.00     | 3                                         | 3.19              | 92.3                            | 33.3                                       | 1.3                                          | 1.0                                                   |
| 20  | 2     | 0.50     | 2                                         | 10.00             | 229.0                           | 42.0                                       | 6.5                                          | 12.5                                                  |
|     |       | 0.75     | 2                                         | 14.42             | 340.5                           | 39.5                                       | 7.5                                          | 13.5                                                  |
|     |       | 1.00     | 3                                         | 3.29              | 220.3                           | 21.7                                       | 2.0                                          | 6.0                                                   |
|     | 5     | 0.00     | 1                                         | 280.01            | 4847.0                          | 48.0                                       | 4.0                                          | 47.0                                                  |
|     |       | 1.00     | 1                                         | 11.53             | 275.0                           | 37.0                                       | 10.0                                         | 4.0                                                   |
|     | 8     | 0.00     | 1                                         | 121.04            | 1926.0                          | 47.0                                       | 12.0                                         | 30.0                                                  |
|     |       | 0.25     | 2                                         | 168.56            | 2541.5                          | 44.0                                       | 21.5                                         | 14.5                                                  |
|     |       | 0.50     | 2                                         | 3.08              | 80.0                            | 38.0                                       | 1.0                                          | 2.0                                                   |
|     |       | 0.75     | 2                                         | 2.81              | 75.5                            | 38.0                                       | 1.0                                          | 1.5                                                   |
|     |       | 1.00     | 2                                         | 2.81              | 75.5                            | 38.0                                       | 1.0                                          | 1.5                                                   |
| 25  | 2     | 0.75     | 1                                         | 41.55             | 733.0                           | 39.0                                       | 2.0                                          | 14.0                                                  |
|     |       | 1.00     | 3                                         | 107.89            | 1582.3                          | 34.7                                       | 4.0                                          | 7.7                                                   |
|     | 5     | 0.75     | 2                                         | 39.85             | 661.5                           | 37.5                                       | 1.0                                          | 2.5                                                   |
|     |       | 1.00     | 2                                         | 39.87             | 661.5                           | 37.5                                       | 1.0                                          | 2.5                                                   |
|     | 8     | 0.75     | 1                                         | 189.46            | 2020.0                          | 49.0                                       | 7.0                                          | 6.0                                                   |
|     |       | 1.00     | 1                                         | 191.31            | 2020.0                          | 49.0                                       | 7.0                                          | 6.0                                                   |

ple [3]). In P4 and P5, the maximum number of effective constraints is about twice what it is in P1 and P2. In the Land-Powell algorithm [2], this number is quite critical: it is equal to the size of the (reduced) inverse basis and is strongly related to the computation time.

## 5. CONCLUSION

We have presented an integer programming formulation and an exact algorithm for an important class of multidepot vehicle routing problems. The formulation includes degree constraints, subtour elimination constraints, chain barring constraints and integrality constraints. The problems are solved by a branch and bound algorithm which initially relaxes the last three types of constraints. Computational results indicate that problems involving up to about 40 nodes can be solved optimally, depending on the values of the parameters considered. This is small compared with the TSP [1] or the  $m$ -TSP [3]. However, to the authors' knowledge, the algorithm described in this paper constitutes the first attempt ever presented in the operational research literature to solve P5 optimally.

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